LARGE SOLUTIONS TO THE p-LAPLACIAN FOR LARGE p

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Abstract. In this work we consider the behaviour for large values of p of the unique positive weak solution u_p to $\Delta_p u = u^q$ in Ω, $u = +\infty$ on $\partial\Omega$, where $q > p - 1$. We take $q = q(p)$ and analyze the limit of u_p as $p \to \infty$. We find that when $q(p)/p \to Q$ the behaviour strongly depends on Q. If $1 < Q < \infty$ then solutions converge uniformly in compacts to a viscosity solution of $\max\{-\Delta_\infty u,\ -|\nabla u| + u^Q\} = 0$ with $u = +\infty$ on $\partial\Omega$. If $Q = 1$ then solutions go to ∞ in the whole Ω and when $Q = \infty$ solutions converge to 1 uniformly in compact subsets of Ω, hence the boundary blow-up is lost in the limit.

1. Introduction.

The aim of the present work is the study of the behaviour of positive weak solutions to the problem

$$\begin{cases} \Delta_p u = u^q & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

for large values of p. In fact, we consider the limit as $p \to \infty$. Here $\Omega \subset \mathbb{R}^N$ is a bounded C^2 domain, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ stands for the p-Laplacian operator with $p > 1$ and $q > p - 1$. The boundary condition is to be understood as $u(x) \to \infty$ as $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0$.

Problems like (1.1) are usually known in the literature as boundary blow-up problems, and their solutions are also named "large solutions". A large amount of work has been dedicated to study such problems. The special case $p = 2$ of (1.1) (or a linear perturbation of it) was considered in [3], [4], [10], [17], [21], [24], or [28], while the general case, $p > 1$, was the subject of [12]. Some more general problems have also been analyzed, when the power reaction is substituted by a smooth increasing function $f(u)$ (see [5], [22]). We refer the reader to [16] for a more complete list of references. However, at the best of our knowledge, the present work seems to be the first one to deal with large solutions with varying p.

On the other hand, the behaviour of solutions as $p \to \infty$ for problems related to (1.1) has also been intensively studied in the natural framework of viscosity solutions. The limit as $p \to \infty$ of solutions to the $p-$Laplacian has several applications, for example, in studying Lipschitz extensions, [2], mass transfer problems, [14], [15], concentration of branches of solutions, [8], etc. For general references, we refer to [7], [15], [19] or [20], and to the recent survey [2]. We remark that in these works the boundary condition was always of Dirichlet type, except in [15], where a nonlinear boundary condition was imposed. We also quote [11], where a boundary blow-up problem with the infinity Laplacian operator was considered.

Returning to problem (1.1), it is well-known that it admits a unique positive weak solution provided that $q > p-1$ (cf. for instance [12]). This solution will be denoted by u_p. Since our intention is to analyze the behaviour of u_p as $p \to \infty$, it will follow that $q \to \infty$ as well. Our main hypothesis will be to assume that $q = q(p)$ is a function of p, while the limit

$$Q = \lim_{p\to\infty} \frac{q(p)}{p}, \tag{1.2}$$

exists (and it will then follow that $Q \geq 1$). Observe that $Q = \infty$ is not excluded. However, we have to point out that our main result, Theorem 1, still holds with weaker assumptions (see Remark 1 (a) below).

To have an insight into the expected behaviour of u_p, we briefly consider problem (1.1) in a one-dimensional situation, namely when $\Omega = (0,\infty)$ is the half-line:

$$\begin{cases} (|u'|^{p-2}u')' = u^q & \text{in } (0,\infty), \\ u(0) = +\infty. \end{cases} \tag{1.3}$$

The unique solution to (1.3) is explicit, and is given by

$$u_p(x) = \left(\alpha_p^{p-1}(\alpha_p+1)(p-1)\right)^{\frac{1}{q-p+1}} x^{-\alpha_p}$$

with $\alpha_p = p/(q-p+1)$. It can be seen then that the following conclusions hold: (i) If $Q = 1$, then $\alpha_p \to \infty$, and so $u_p \to \infty$ uniformly on compact subsets of $(0,\infty)$; (ii) If $1 < Q < \infty$, $u_p \to \alpha_0^{\alpha_0} x^{-\alpha_0}$, uniformly on compacts of $(0,\infty)$, where $\alpha_0 = 1/(Q-1)$; (iii) If $Q = \infty$, then $u_p \to 1$ uniformly in compact subsets of $(0,\infty)$ (since $\alpha_p \to 0$).

Our main objective will be to prove that the same features are valid for problem (1.1) in a smooth bounded domain Ω of $\mathbb{R}^N$. As is to be expected, there is not an explicit expression for the solutions anymore, and thus the study is not so simple.

Now we proceed with the statement of our results.

Theorem 1. *For $p > 1$, let u_p be the unique solution to* (1.1) *with $q = q(p) > p-1$, and assume $q(p)/p \to Q$ as $p \to \infty$. Then we have:*

(1) *If $Q = 1$, then u_p converges uniformly to $+\infty$ in Ω as $p \to \infty$.*

(2) *If $1 < Q < \infty$, then u_p converges uniformly on compact subsets of Ω to a viscosity solution u to*

$$\begin{cases} \max\{-\Delta_\infty u,\ -|\nabla u| + u^Q\} = 0 & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

which verifies $u \in C^\gamma(\Omega)$ for every $\gamma \in (0,1)$. Moreover, $u(x) \leq \alpha_0^{\alpha_0} d(x)^{-\alpha_0}$ in Ω, and there exists $\delta > 0$ such that

$$u(x) = \alpha_0^{\alpha_0} d(x)^{-\alpha_0} \quad \text{if } 0 < d(x) < \delta, \tag{1.5}$$

with $\alpha_0 = 1/(Q-1)$. Furthermore, u is the only solution to (1.4) *which verifies*

$$u(x) \sim \alpha_0^{\alpha_0} d(x)^{-\alpha_0} \quad \text{as } d(x) \to 0. \tag{1.6}$$

(3) *If $Q = \infty$, then u_p converges uniformly on compact subsets of Ω to $u = 1$. Therefore the limit loses the explosive behaviour on the boundary.*

Remarks 1. (a) It is worthy of mention that Theorem 1 is still valid even if a weaker version of (1.2) holds. That is, if $p_n \to \infty$ and $q_n > p_n - 1$ are arbitrary sequences such that $q_n/p_n \to Q$, then the same assertions hold. We refer to the proofs in Section 4.

(b) It is already known that $u_p \sim A_p\, d(x)^{-\alpha_p}$ as $d(x) \to 0$ (cf. [12]), where $\alpha_p = p/(q-p+1)$ and $A_p = [\alpha_p^{p-1}(\alpha_p+1)(p-1)]^{1/(q-p+1)}$. Since, in the case $0 < Q < 1$, $\alpha_p \to \alpha_0$ while $A_p \to \alpha_0^{\alpha_0}$, we deduce from (1.6) that the behaviour near the boundary of the limit u is the limit of the behaviour near the boundary of u_p.

(c) It is a consequence of (2) in Theorem 1 that the function $\alpha_0^{\alpha_0} d(x)^{-\alpha_0}$ is always a solution to (1.4) in a neighbourhood of $\partial\Omega$. However, it is not expected to be a solution in Ω, since this will deeply depend on the geometric properties of the domain. In the case when Ω is a ball or an annulus, it is not hard to show that this is indeed the unique solution (see Remark 2 and Theorem 2 in Section 2).

To finish the introduction let us briefly describe the ideas and methods used to prove Theorem 1. The key to deduce all behaviours is to analyze in detail problem (1.1) in a ball $B(x_0, R)$ of $\mathbb{R}^N$ where solutions are radial, obtaining the explicit dependence on p of these estimates. Then by means of comparison arguments we will obtain the desired results in smooth bounded domains Ω. We stress that in the case $1 < Q < \infty$ it is not too hard to pass to the limit (through subsequences) and obtain a positive viscosity solution u to (1.4). However, to deduce that the limit as $p \to \infty$ exists, we need to prove the uniqueness of positive solutions to (1.4) verifying (1.6). This uniqueness is not a consequence of previous results on viscosity solutions (see for instance [6], [9], [19], [20] or [15]), and although we are using some ideas from the general maximum principle in [9], the proof is not straightforward. We refer to Section 3 for the details. We mention in passing that the regularity of the limit u in part (2) of Theorem 1 is not a consequence of a regularity theory for equations like (1.4). Indeed it is really hard to obtain general regularity results, as some existing literature shows (see [26]).

The paper is organized as follows: in Section 2, we obtain some preliminary results in the case where Ω is a ball in $\mathbb{R}^N$. Section 3 deals with viscosity solutions, containing in particular the essential uniqueness theorem of solutions to (1.4) verifying the condition (1.6). Finally, Section 4 is dedicated to the proof of Theorem 1.

2. Preliminaries on radial solutions

In this section we perform a preliminary analysis of problem (1.1) in the particular case where Ω is a ball of $\mathbb{R}^N$, denoted by $B(x_0, R)$. With no loss of generality, we may assume throughout that $x_0 = 0$ (we are not setting however $R = 1$). Our intention is to prove the following weaker version of Theorem 1.

Theorem 2. *Let u_p be the unique positive solution to* (1.1) *in* $B(0, R)$. *Assume $q/p \to Q$ when $p, q \to \infty$. Then:*

(1) *If* $Q = 1$, *then for every* $\delta \in (0, R)$, $u_p \to \infty$ *uniformly in* $\delta \le |x| < R$ *as* $p \to \infty$.
(2) *If* $1 < Q < \infty$, *then for every* $\delta > 0$, $u_p(x) \to \alpha_0^{\alpha_0}(R - |x|)^{-\alpha_0}$ *uniformly in* $\delta \le |x| \le R - \delta$, *where* $\alpha_0 = 1/(Q - 1)$.
(3) *If* $Q = \infty$, *then for every* $\delta \in (0, R)$, $u_p \to 1$ *uniformly in* $\delta \le |x| \le R - \delta$.

Remark 2. We quote that the function $u = \alpha_0^{\alpha_0}(R - |x|)^{-\alpha_0}$ is a solution to the equation

$$\max\{-\Delta_\infty u, -|\nabla u| + u^Q\} = 0$$

in $B(0, R)$. Indeed, it verifies $|\nabla u| = u^Q$, $\Delta_\infty u > 0$ in $B(0, R) \setminus \{0\}$ in the classical sense, and in the center of the ball in the viscosity sense (cf. [20] for a related situation).

We now remark that, when $\Omega = B(0, R)$, the uniqueness and regularity of u_p imply that it is radial, that is, $u_p(x) = u_p(|x|)$. Hence it is well known that the solution has to satisfy the ordinary differential equation:

$$\begin{cases} (r^{N-1}\varphi_p(u'))' = r^{N-1}u^q & \text{in } (0, R) \\ u'(0) = 0, \qquad u(R) = \infty, \end{cases}$$

where $'$ stands for derivative with respect to $r = |x|$, and $\varphi_p(z) = |z|^{p-2}z$. We begin with two basic lemmas, which provide us with precise estimates of the solutions in terms of p and q. For their statements, we need to introduce the function

$$I(p, q) = \int_1^\infty \frac{dz}{(z^{q+1} - 1)^{1/p}}, \tag{2.1}$$

which will play a fundamental role in the proof of Theorem 2. We also denote by p' the Hölder conjugate of p, i.e. $p' = p/(p - 1)$. Then:

Lemma 3. *Assume* $p > N$. *Then for every* $\delta \in (0, R)$, *we have*

$$u_p(x) \le \left(\frac{q+1}{p'}\left(\frac{R}{\delta}\right)^{(N-1)p'}\right)^{\frac{1}{q-p+1}} I(p, q)^{\frac{p}{q-p+1}}(R - |x|)^{-\alpha_p}, \tag{2.2}$$

for $\delta \le |x| < R$, *where* $\alpha_p = p/(q - p + 1)$.

Proof. We introduce the change of variables

$$s = \frac{1}{1 - \gamma}(R^{1-\gamma} - r^{1-\gamma}),$$

where $\gamma = (N - 1)/(p - 1) < 1$. If we denote $v(s) = u(r)$ (the subindex p will be dropped along this proof to simplify the notation), we find that it satisfies the non-autonomous one-dimensional problem:

$$\begin{cases} \varphi_p(v')' = g_p(s)v^q & \text{in } (0, T) \\ v(0) = \infty, \qquad v'(T) = 0, \end{cases} \tag{2.3}$$

where $T = R^{1-\gamma}/(1 - \gamma)$ and $g_p(s) = r^{(N-1)p'}$. We notice that equation (2.3) implies that $v' < 0$ in $(0, T)$. Thus if we multiply by v', we arrive at

$$\varphi_p(v')'v' \le \delta^{(N-1)p'}v^q v' \qquad \text{in } (0, T_\delta) \tag{2.4}$$

for $r \geq \delta$, where $T_\delta = (R^{1-\gamma} - \delta^{1-\gamma})/(1-\gamma)$. Now let x, y be such that $0 < x < y < T_\delta$. If we integrate equation (2.4) in the interval (x, y), we obtain the inequality

$$|v'(x)|^p \geq \frac{p'}{q+1}\delta^{(N-1)p'}(v(x)^{q+1} - v(y)^{q+1}),$$

where a term $|v'(y)|^p$, which appears after the integration has been dropped. Since $v' < 0$, we obtain after integrating with respect to x in $(0, y)$ that

$$\int_{v(y)}^{\infty} \frac{d\tau}{(\tau^{q+1} - v(y)^{q+1})^{1/p}} \geq \left(\frac{p'}{q+1}\delta^{(N-1)p'}\right)^{1/p} y,$$

provided $y \leq T_\delta$. Letting $\tau = v(y)z$ in the integral, we arrive at

$$v(y) \leq \left(\frac{q+1}{p'}\delta^{-(N-1)p'}\right)^{\frac{1}{q-p+1}} I(p,q)^{\frac{p}{q-p+1}} y^{-\alpha_p},$$

for $y \leq T_\delta$. This implies for u

$$u(x) \leq \left(\frac{q+1}{p'}\delta^{-(N-1)p'}\right)^{\frac{1}{q-p+1}} I(p,q)^{\frac{p}{q-p+1}} \left(\frac{1}{1-\gamma}(R^{1-\gamma} - |x|^{1-\gamma})\right)^{-\alpha_p},$$

whenever $\delta \leq |x| < R$, and (2.2) follows once we note that $R^{1-\gamma} - |x|^{1-\gamma} \geq (1-\gamma)R^{-\gamma}(R - |x|)$. This completes the proof. □

With a similar argument as in the proof of Lemma 3, we can also obtain a lower estimate for the solutions. There is still an alternative way of doing it, which does not use the radial symmetry of the solution: constructing a suitable subsolution.

Lemma 4. *Assume $p > N$. Then for every $\delta \in (0, R)$ we have*

$$u_p(x) \geq \alpha_p^{\alpha_p}\left(\frac{q+1}{p'}\left(\frac{\delta}{R}\right)^{(N-1)p'}\right)^{\frac{1}{q-p+1}} (R - |x|)^{-\alpha_p} \tag{2.5}$$

if $\delta \leq |x| < R$, where $\alpha_p = p/(q-p+1)$.

Proof. We proceed as in Lemma 3. Now notice that, after the change of variables introduced there, we have for the function v,

$$\varphi_p(v')'v' \geq R^{(N-1)p'} v^q v' \qquad \text{in } (0, T). \tag{2.6}$$

Now choose $0 < x < y < T$, and integrate (2.6) in the interval (x, T). We arrive at

$$|v'(x)|^p \leq \frac{p'}{q+1} R^{(N-1)p'} v(x)^{q+1},$$

for $0 < x < y$. Taking into account once again that $v' < 0$, we obtain after integrating with respect to x in $(0, y)$ that

$$v(y) \geq \alpha_p^{\alpha_p}\left(\frac{p'}{q+1}R^{-(N-1)p'}\right)^{\frac{1}{q-p+1}} y^{-\alpha_p}$$

for $0 < y < T$. In the original variable x, this will imply for the function u,

$$u(x) \geq \alpha_p^{\alpha_p}\left(\frac{q+1}{p'}R^{-(N-1)p'}\right)^{\frac{1}{q-p+1}} \left(\frac{1}{1-\gamma}(R^{1-\gamma} - |x|^{1-\gamma})\right)^{-\alpha_p}.$$

The estimate (2.5) is obtained from $R^{1-\gamma} - |x|^{1-\gamma} \leq (1-\gamma)\delta^{-\gamma}(R-|x|)$ when $\delta \leq |x| < R$. □

Before we can finally proceed with the proof of Theorem 2, we need another property of the function $I(p,q)$ given by (2.1).

Lemma 5. *Let $I(p,q)$ be the function defined by* (2.1) *for $p>1$, $q>p-1$. Then if $q/p \to Q > 1$ as $p,q\to\infty$ ($Q=\infty$ is not excluded), we have that*

$$I(p,q)^{\frac{p}{q-p+1}} \to \begin{cases} \left(\dfrac{1}{Q-1}\right)^{\frac{1}{Q-1}} & \text{if } Q<\infty \\ 1 & \text{if } Q=\infty. \end{cases}$$

Proof. Performing the change of variable $z = t^{-1/(q+1)}$ in the integral defining $I(p,q)$, we obtain that

$$I(p,q) = \frac{1}{q+1} B\left(\frac{1}{p} - \frac{1}{q+1}, 1-\frac{1}{p}\right),$$

where B stands for Euler's Beta function. Thus according to well known properties of B,

$$I(p,q) = \frac{1}{q+1}\Gamma\left(\frac{1}{p}-\frac{1}{q+1}\right)\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(1-\frac{1}{q+1}\right)^{-1},$$

where Γ is Euler's Gamma function (see for instance [23]). Thus for $p,q\to\infty$ we have

$$I(p,q)^{\frac{p}{q-p+1}} \sim \left(\frac{1}{q+1}\Gamma\left(\frac{1}{p}-\frac{1}{q+1}\right)\right)^{\frac{p}{q-p+1}}.$$

Now we use that the Gamma function admits an analytic continuation as a meromorphic function in $\mathbb{C}$, with simple poles at $z=-n$, $n=0,1,\ldots$, with corresponding residues $(-1)^n/n!$. In particular, $z\Gamma(z)\to 1$ when $z\to 0$, $z\in\mathbb{C}$ (see Section 1.1 in [23]). This automatically implies that

$$I(p,q)^{\frac{p}{q-p+1}} \sim \left(\frac{p}{q-p+1}\right)^{\frac{p}{q-p+1}},$$

and the conclusion of the lemma follows easily. □

Now we prove Theorem 2.

Proof of Theorem 2. The proof follows by combining estimates (2.2) and (2.5) with Lemma 5 and the evaluation of some limits.

Part 1 ($Q=1$). It follows from Lemma 4 that for $|x|\geq\delta$:

$$u_p(x) \geq \alpha_p^{\alpha_p}\left(\frac{q+1}{p'}\left(\frac{\delta}{R}\right)^{(N-1)p'}\right)^{\frac{1}{q-p+1}} (R-\delta)^{-\alpha_p}.$$

Since for sufficiently large p the term between the big brackets is greater than one, we have that

$$u_p(x) \geq \left(\frac{\alpha_p}{R-\delta}\right)^{\alpha_p},$$

which implies that $u_p\to\infty$ uniformly in $\delta\leq|x|<R$.

Part 2 ($1 < Q < \infty$). Thanks to (2.2), for every $\delta > 0$, we have

$$u_p(x) \le \left(\frac{q+1}{p'} \left(\frac{R}{\delta} \right)^{(N-1)p'} \right)^{\frac{1}{q-p+1}} I(p,q)^{\frac{p}{q-p+1}} (R-|x|)^{-\alpha_p}, \tag{2.7}$$

provided that $\delta \le |x| < R$. Notice that the term between the big brackets converges to 1, while $I(p,q)^{\frac{p}{q-p+1}} \to \alpha_0^{\alpha_0}$, thanks to Lemma 5. Since $\alpha_p \to \alpha_0$, we arrive at

$$\limsup_{p\to\infty} u_p(x) \le \alpha_0^{\alpha_0} (R-|x|)^{-\alpha_0}.$$

On the other hand, the lower estimate provided by Lemma 4 reads as:

$$u_p(x) \ge \alpha_p^{\alpha_p} \left(\frac{q+1}{p'} \left(\frac{\delta}{R} \right)^{(N-1)p'} \right)^{\frac{1}{q-p+1}} (R-|x|)^{-\alpha_p},$$

and since the term between the big brackets also converges to 1, we obtain the desired lower limit:

$$\liminf_{p\to\infty} u_p(x) \ge \alpha_0^{\alpha_0} (R-|x|)^{-\alpha_0}.$$

Finally, it is clear from the estimates that the convergence is uniform in subsets of the form $\delta \le |x| \le R-\delta$.

Part 3 ($Q = \infty$). We first observe that in this case $\alpha_p \to 0$. Choose $\delta \in (0,R)$; thanks to Lemma 3, we have

$$u_p(x) \le \left(\frac{q+1}{p'} \left(\frac{R}{\delta} \right)^{(N-1)p'} \right)^{\frac{1}{q-p+1}} I(p,q)^{\frac{p}{q-p+1}} \delta^{-\alpha_p}, \tag{2.8}$$

for $\delta \le |x| \le R-\delta$. If we now let $p \to \infty$ in (2.8) and use Lemma 5, we obtain

$$\limsup_{p\to\infty} u_p(x) \le 1 \tag{2.9}$$

uniformly for $\delta \le |x| \le R-\delta$.

On the other hand, thanks to Lemma 4, we have for $|x| \ge \delta$ that

$$u_p(x) \ge \alpha_p^{\alpha_p} \left(\frac{q+1}{p'} \left(\frac{\delta}{R} \right)^{(N-1)p'} \right)^{\frac{1}{q-p+1}} R^{-\alpha_p}.$$

We only need to let $p \to \infty$ to arrive at

$$\liminf_{p\to\infty} u_p(x) \ge 1$$

uniformly for $|x| \ge \delta$, which, together with (2.9) proves the theorem. □

3. Some remarks on viscosity solutions.

Since our intention is letting $p \to \infty$ in (1.1), the concept of weak solution is not an appropriate one. In fact, the underlying space for weak solutions is $W^{1,p}_{\mathrm{loc}}(\Omega)$, which varies with p. Thus we will rather use the concept of viscosity solution. We begin by recalling a standard definition (see [9]). In what follows, we denote by $S(N)$ the space of $N \times N$ symmetric matrices, and by D^2u the Hessian Matrix of u.

Definition 6. *Let $F : \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ be a given function. A lower (resp. upper) semi-continuous function u is a viscosity supersolution (resp. subsolution) to*

$$F(u, \nabla u, D^2u) = 0 \qquad \text{in } \Omega \tag{3.1}$$

if for every $\phi \in C^2(\Omega)$ such that $u - \phi$ has a strict minimum (resp. maximum) at the point $x_0 \in \Omega$ with $u(x_0) = \phi(x_0)$ we have:

$$F(\phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \geq 0 \ (\textit{resp.} \leq 0).$$

Finally, u is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.

We remark that it is not necessary to require that the minimum (resp. maximum) in this definition is strict. An equivalent definition is obtained when the word "strict" is dropped.

Before proceeding further, we need to check that the unique weak solution u_p to (1.1) is indeed a viscosity solution. Thus, in the light of Definition 6, we say that an upper semi-continuous function u is a viscosity supersolution to (1.1) if, for every $\phi \in C^2(\Omega)$ such that $u - \phi$ has a strict minimum at $x_0 \in \Omega$ with $u(x_0) = \phi(x_0)$, we have

$$\Delta_p\phi(x_0) = (p-2)|\nabla\phi|^{p-4}\Delta_\infty\phi(x_0) + |\nabla\phi|^{p-2}\Delta\phi(x_0) \leq \phi^q(x_0)$$

(to avoid complications, and since we are interested in large p, we may always assume $p > 2$). Here $\Delta_\infty u$ represents the ∞-Laplacian operator, given by $\Delta_\infty u = \nabla u D^2u \nabla u^T$.

We have the following result. Although the proof follows by similar arguments as those in Lemma 2.3 of [15] or Lemma 1.8 in [20], we include it here for completeness.

Lemma 7. *Let u_p be the unique weak solution to* (1.1) *for $p > 2$. Then u_p is a viscosity solution of*

$$\begin{cases} \Delta_p u = u^q & \text{in } \Omega \\ u = \infty & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Proof. We first recall that, according to interior regularity for the p-Laplace equation, $u_p \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ (see [13] and [27]). In particular, u is continuous in Ω. We are only showing that u_p is a viscosity supersolution, since the proof that it is a viscosity subsolution is entirely similar. For the sake of brevity we are dropping the subindex p in what follows.

Assume u is not a viscosity supersolution. Then there exist a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u - \phi$ has a strict minimum at x_0 with

$u(x_0) = \phi(x_0)$, but $\Delta_p\phi(x_0) > \phi^q(x_0)$. Since $u(x_0) = \phi(x_0)$, thanks to continuity there exists $r > 0$ so that for x in the ball $B(x_0, r)$ we have

$$\Delta_p\phi(x) > u^q(x). \tag{3.3}$$

Now set $m = \inf_{|x-x_0|=r}(u - \phi)$. By diminishing r if necessary, and since x_0 is a minimum for $u - \phi$, we have $m > 0$. Let $\psi(x) = \phi(x) + m/2$. It is then clear that the function $(\psi - u)^+$ belongs to $W_0^{1,p}(B(x_0, r))$, and it is nontrivial since $\psi(x_0) = u(x_0) + m/2 > u(x_0)$. Thus if we multiply (3.3) by $(\psi - u)^+$ and integrate in $B(x_0, r)$, we arrive at

$$\int_{\{\psi>u\}\cap B(x_0,r)} |\nabla\psi|^{p-2}\nabla\psi\nabla(\psi - u) < -\int_{\{\psi>u\}\cap B(x_0,r)} (\psi - u)u^q. \tag{3.4}$$

On the other hand, if we take $(\psi-u)^+$ (extended to be zero outside $B(x_0, r)$) as a test function in the weak formulation of (1.1), we get

$$\int_{\{\psi>u\}\cap B(x_0,r)} |\nabla u|^{p-2}\nabla u\nabla(\psi - u) = -\int_{\{\psi>u\}\cap B(x_0,r)} (\psi - u)u^q \tag{3.5}$$

Hence, subtracting (3.4) and (3.5) and using the monotonicity of the p-Laplacian for $p > 2$ (Lemma 1 in [27]):

$$\begin{aligned} C(N,p)&\int_{\{\psi>u\}\cap B(x_0,r)} |\nabla\psi - \nabla u|^p \\ &\le \int_{\{\psi>u\}\cap B(x_0,r)} (|\nabla\psi|^{p-2}\nabla\psi - |\nabla u|^{p-2}\nabla u)\nabla(\psi - u) < 0, \end{aligned}$$

for a positive constant $C(N, p)$, a contradiction. This shows that u is a viscosity supersolution, and concludes the proof of the lemma. □

Now we consider the expected limit problem as $p \to \infty$ of the solutions u_p (at least in the case $1 < Q < \infty$), namely

$$\begin{cases} \max\{-\Delta_\infty u,\ -|\nabla u| + u^Q\} = 0 & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $Q > 1$. This problem admits a unique viscosity solution u with the boundary behaviour given by (1.6). This will be of particular interest in the proof of Theorem 1. Our intention is to apply the general results in [9], but we remark that the comparison principle there (Theorem 3.3) is not directly applicable since the equation in (1.4) does not satisfy their condition (3.13); that is, if G_∞ denotes the left-hand side of (1.4) then there does not exist a positive constant γ so that

$$G_\infty(z, \xi, X) - G_\infty(w, \xi, X) \ge \gamma(z - w)$$

whenever $z \ge w$, $\xi \in \mathbb{R}^N$, $X \in S(N)$ (we point out that G_∞ is not even strictly increasing in the variable z). Thus we proceed differently and, following [20], we perform a change of variables in our equation.

Lemma 8. *Let u be a continuous positive viscosity solution to* (1.4). *Then $v = u^{1/2}$ is a positive viscosity solution to*

$$\begin{cases} \max\left\{-\Delta_\infty v - \dfrac{|\nabla v|^4}{v},\ -2|\nabla v| + v^{2Q-1}\right\} = 0 & \textit{in } \Omega \\ v = +\infty & \textit{on } \partial\Omega. \end{cases} \tag{3.6}$$

Proof. We only prove that v is a viscosity subsolution, the proof that it is a supersolution being similar (of course the boundary condition still holds for v). Let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $v - \phi$ has a strict maximum at x_0, with $v(x_0) = \phi(x_0)$. Then ϕ^2 is a valid test function in (1.4) ($u - \phi^2$ attains a maximum at x_0), and we deduce that

$$\max\{-\Delta_\infty \phi^2(x_0), -|\nabla \phi^2(x_0)| + \phi^{2Q}(x_0)\} \le 0.$$

After some manipulations, we see that this leads to

$$\max\{-8\phi^3 \Delta_\infty \phi(x_0) - 8\phi^2 |\nabla \phi(x_0)|^4, -2\phi(x_0)|\nabla \phi(x_0)| + \phi(x_0)^{2Q}\} \le 0,$$

which implies, as $\phi(x_0) = u(x_0)^{1/2} > 0$, that v is a subsolution of (3.6). □

We are next showing that, if v is a viscosity supersolution to (3.6), then $(1+\varepsilon)v + \varepsilon$ is a strict viscosity supersolution, in the sense that for every $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $(1+\varepsilon)v + \varepsilon - \phi$ has a strict minimum at x_0 with $(1+\varepsilon)v(x_0) + \varepsilon = \phi(x_0)$, then

$$\max\left\{-\Delta_\infty \phi(x_0) - \frac{|\nabla \phi(x_0)|^4}{\phi(x_0)},\ -2|\nabla \phi(x_0)| + \phi(x_0)^{2Q-1}\right\} > 0.$$

This will be essential in order to apply the general results in [9].

Lemma 9. *Let $v \in C(\Omega)$ be a continuous, positive viscosity supersolution to* (3.6)*. Then for every $\varepsilon > 0$, $(1+\varepsilon)v + \varepsilon$ is a strict viscosity supersolution.*

Proof. Let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that $(1+\varepsilon)v + \varepsilon - \phi$ has a strict minimum at x_0 with $(1+\varepsilon)v(x_0) + \varepsilon = \phi(x_0)$. Then $(\phi - \varepsilon)/(1+\varepsilon)$ is a valid test function in (3.6), and we have

$$\max\left\{-\Delta_\infty \phi(x_0) - \frac{|\nabla \phi(x_0)|^4}{\phi(x_0) - \varepsilon},\right.$$
$$\left. -2|\nabla \phi(x_0)| + \left(\frac{1}{1+\varepsilon}\right)^{2(Q-1)} (\phi(x_0) - \varepsilon)^{2Q-1}\right\} \ge 0.$$

Assume first that

$$-2|\nabla \phi(x_0)| + \left(\frac{1}{1+\varepsilon}\right)^{2(Q-1)} (\phi(x_0) - \varepsilon)^{2Q-1} \ge 0. \tag{3.7}$$

Then we get

$$-2|\nabla \phi(x_0)| + \phi(x_0)^{2Q-1} \ge \phi(x_0)^{2Q-1} - \left(\frac{1}{1+\varepsilon}\right)^{2(Q-1)} (\phi(x_0) - \varepsilon)^{2Q-1} > 0.$$

If, on the contrary, (3.7) does not hold, then

$$-\Delta_\infty \phi(x_0) - \frac{|\nabla \phi(x_0)|^4}{\phi(x_0) - \varepsilon} \ge 0,$$

and $|\nabla \phi(x_0)| > 0$, thanks to the reversed inequality in (3.7). This implies

$$-\Delta_\infty \phi(x_0) - \frac{|\nabla \phi(x_0)|^4}{\phi(x_0)} \ge \frac{|\nabla \phi(x_0)|^4}{\phi(x_0) - \varepsilon} - \frac{|\nabla \phi(x_0)|^4}{\phi(x_0)} > 0,$$

as was to be proved. □

Remark 3. It is not hard to show that Lemma 9 does not necessarily hold for the original problem (1.4). That is, $(1+\varepsilon)v+\varepsilon$ is a supersolution whenever v is, but it need not be strict.

We finally prove that we can take advantage of the general framework of Section 3 in [9] to prove uniqueness of solutions to problem (1.4). To this end, we denote for $z \in \mathbb{R}^+$, $\xi \in \mathbb{R}^N$, $X \in S(N)$

$$F_\infty(z,\xi,X) = \max\left\{-\xi X \xi^T - \frac{|\xi|^4}{z},\ -2|\xi| + z^{2Q-1}\right\}. \tag{3.8}$$

Hence, (3.6) reads as $F_\infty(u,\nabla u, D^2u) = 0$. We remark that this equation is proper (or degenerate elliptic) in the terminology of [9]. In fact, it verifies

$$F_\infty(z,\xi,X) \geq F_\infty(z,\xi,Y),$$

whenever $X \leq Y$ (i.e. $Y - X$ is semidefinite positive) and

$$F_\infty(z,\xi,X) \leq F_\infty(w,\xi,X),$$

whenever $z \leq w$. Under these conditions, we have the following important theorem.

Theorem 10. *Let $u_1, u_2 \in C(\Omega)$ be strictly positive viscosity solutions to* (1.4)*, with*

$$\lim_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} = 1.$$

Then $u_1 = u_2$ in Ω.

Proof. Observe that it suffices to prove that if v_1 and v_2 are positive viscosity solutions to (3.6) with $v_1/v_2 \to 1$ as $d(x) \to 0$ then $v_1 = v_2$. For this sake, we adapt an argument in [17].

Choose $\varepsilon > 0$. Then there exists $\delta > 0$ such that $v_1 \leq (1+\varepsilon)v_2 < (1+\varepsilon)v_2 + \varepsilon$ in $\Omega \setminus \Omega_\delta$, where $\Omega_\delta = \{x \in \Omega :\ d(x) > \delta\}$. Thanks to Lemma 9, $(1+\varepsilon)v_2+\varepsilon$ is a strict viscosity supersolution to (3.6) in Ω_δ. We claim that $v_1 \leq (1+\varepsilon)v_2 + \varepsilon$ in Ω_δ. Indeed, assume on the contrary that $v_1(z) > (1+\varepsilon)v_2+\varepsilon(z)$ for some $z \in \Omega_\delta$. For $\sigma > 0$, we consider the function

$$v_\sigma(x,y) = v_1(x) - (1+\varepsilon)v_2(y) - \varepsilon - \frac{\sigma}{2}|x-y|^2,$$

defined in $\overline{\Omega}_\delta \times \overline{\Omega}_\delta$. It follows that there exist points $(x_\sigma, y_\sigma) \in \overline{\Omega}_\delta \times \overline{\Omega}_\delta$ such that the function v_σ attains its maximum in $\overline{\Omega}_\delta \times \overline{\Omega}_\delta$ at (x_σ, y_σ). For a sequence $\sigma_n \to \infty$, we may assume that $x_{\sigma_n} \to x_0 \in \overline{\Omega}_\delta$. Thanks to Lemma 3.1 in [9], x_0 is a maximum point of $v_1 - (1+\varepsilon)v_2 - \varepsilon$, and $y_{\sigma_n} \to x_0$. We now apply Theorem 3.2 in [9] (see also the discussion after it) to obtain two functions $\phi, \psi \in C^2(\Omega)$ such that $v_1 - \phi$ has a maximum at x_{σ_n}, $\nabla\phi(x_{\sigma_n}) = \sigma_n(x_{\sigma_n} - y_{\sigma_n})$, $D^2\phi(x_{\sigma_n}) = X_{\sigma_n}$ and $(1+\varepsilon)v_2 + \varepsilon - \psi$ has a minimum at y_{σ_n} with $\nabla\psi(y_{\sigma_n}) = \sigma_n(x_{\sigma_n} - y_{\sigma_n})$, $D^2\psi(y_{\sigma_n}) = Y_{\sigma_n}$. Moreover, the matrices $X_{\sigma_n}, Y_{\sigma_n} \in S(N)$ verify $X_{\sigma_n} \leq Y_{\sigma_n}$. Thanks to the monotonicity properties of F_∞ quoted before and the fact that v_1 is a viscosity subsolution while $(1+\varepsilon)v_2 + \varepsilon$ is a strict viscosity supersolution, we arrive at

$$\begin{aligned} 0 &\geq F_\infty(v_1(x_{\sigma_n}), \sigma_n(x_{\sigma_n} - y_{\sigma_n}), X_{\sigma_n}) \\ &\geq F_\infty((1+\varepsilon)v_2(y_{\sigma_n} + \varepsilon), \sigma_n(x_{\sigma_n} - y_{\sigma_n}), Y_{\sigma_n}) > 0, \end{aligned}$$

a contradiction.

Thus we have shown that $v_1 \le (1+\varepsilon)v_2 + \varepsilon$ in Ω_δ. Taking into account the way δ was selected, we have $v_1 \le (1+\varepsilon)v_2 + \varepsilon$ in Ω. Letting $\varepsilon \to 0$, we obtain $v_1 \le v_2$, and the symmetric argument proves $v_1 = v_2$. □

4. Proof of Theorem 1.

We are dedicating this section to the proof of Theorem 1. To clarify the exposition, we divide the proof in several lemmas. We begin by considering the simpler cases where $Q = 1$ or $Q = \infty$. The key of the proofs is to compare the solutions with suitable radial solutions, which have been studied in Section 2.

Lemma 11. *Assume $Q = 1$, then $u_p \to \infty$ uniformly in Ω as $p \to \infty$.*

Proof. Assume with no loss of generality that $0 \notin \Omega$. Choose $R > 0$ such that $\Omega \subset\subset B(0,R)$, and consider problem (1.1) in $B(0,R)$:

$$\begin{cases} \Delta_p u = u^q & \text{in } B(0,R) \\ u = +\infty & \text{on } \partial B(0,R). \end{cases} \tag{4.1}$$

Let $u_{p,B}$ be the unique solution to (4.1). As $u_{p,B}$ is finite on $\partial\Omega$, we get by comparison that

$$u_p(x) \ge u_{p,B}(x) \quad x \in \Omega. \tag{4.2}$$

We now choose $\delta > 0$ so that $\Omega \subset B(0,R)\setminus B(0,\delta)$. According to Theorem 2, part 1, we have that $u_{p,B} \to \infty$ uniformly in compacts of $B(0,R) \setminus B(0,\delta)$, and thus (4.2) implies that $u_p \to \infty$ uniformly in Ω. This concludes the proof. □

Lemma 12. *Assume $Q = \infty$, then u_p converges uniformly on compact subsets of Ω to $u = 1$.*

Proof. First, observe that the comparison (4.2) in the proof of Lemma 11 remains valid, where $u_{p,B}$ is the unique solution to (4.1), and R, δ are chosen to have $\Omega \subset B(0,R) \setminus B(0,\delta)$. Hence, thanks to Theorem 2, part 3, we get

$$\liminf_{p\to\infty} u_p(x) \ge 1$$

uniformly in Ω.

To prove the complementary upper estimate, let $K \subset \Omega$ be a compact set. Then there exist points $x_1, \dots, x_m$ and positive numbers $R_1, \dots, R_m$ such that

$$K \subset \bigcup_{j=1}^{m} \left(B(x_j, 2R_j) \setminus \overline{B(x_j, R_j)} \right),$$

while $B(x_j, 3R_j) \subset \Omega$. For every j, $1 \le j \le m$, consider the auxiliary problem

$$\begin{cases} \Delta_p u = u^q & \text{in } B(x_j, 3R_j) \\ u = +\infty & \text{on } \partial B(x_j, 3R_j), \end{cases} \tag{4.3}$$

which has a unique positive solution $u_{p,j}$. Since u_p is finite on $\partial B(x_j, 3R_j)$, it follows by comparison that $u_p \le u_{p,j}$ in $B(x_j, 3R_j)$. According to Theorem

2, part 3, we have $u_{p,j} \to 1$ as $p \to \infty$, uniformly in $\overline{B(x_j, 2R_j)} \setminus B(x_j, R_j)$. Then, it is easy to deduce that

$$\limsup_{p\to\infty} u_p(x) \le 1,$$

uniformly in K. This completes the proof. $\square$

We finally consider the more difficult – and of course more interesting – case in which $1 < Q < \infty$. We begin by showing that we can indeed pass to the limit as $p \to \infty$.

Lemma 13. *Assume $1 < Q < \infty$. Then for every sequence $p_n \to \infty$, there exists a subsequence (still denoted by $\{p_n\}$) and a strictly positive function $u \in C^\gamma(\Omega)$ for every $\gamma \in (0,1)$, such that $u_{p_n} \to u$ uniformly on compacts of Ω. Moreover:*

$$u(x) \le \alpha_0^{\alpha_0} d(x)^{-\alpha_0} \tag{4.4}$$

in Ω, where $\alpha_0 = 1/(Q-1)$.

Proof. We begin by noting that the proof of the upper estimate in Lemma 12 is still valid now, and thus, for every compact K, we obtain thanks to Theorem 2, part 2 (we follow the notation of Lemma 12):

$$\limsup_{p\to\infty} u_p(x) \le \alpha_0^{\alpha_0}(3R_j - |x - x_j|)^{-\alpha_0}, \text{ for } x \in \overline{B(x_j, 2R_j)} \setminus B(x_j, R_j).$$

This in particular shows that the sequence u_p is uniformly bounded in K.

We are next showing that it is possible to obtain $W^{1,p}_{\text{loc}}(\Omega)$ bounds for the solutions u_p, independently of p. We remark that the usual strategies (cf. for instance [15]) cannot be employed since $u_p \notin W^{1,p}(\Omega)$. Thus let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ be smooth subdomains, and choose $\phi \in C_0^\infty(\Omega'')$ such that $\phi \ge 0$ and $\phi > M$ in Ω', where M is an upper bound for u_p in Ω''. If we take as a test function in the weak formulation of (1.1) the function $(\phi - u_p)^+$ (notice that $(\phi - u_p)^+ \in W_0^{1,p}(\Omega'')$), we obtain:

(4.5)

$$\begin{aligned}\int_{\phi>u_p} |\nabla u_p|^p &= \int_{\phi>u_p} |\nabla u_p|^{p-2}\nabla u_p \nabla\phi + \int_{\phi>u_p} u_p^q(\phi - u_p)\\ &\le \left(\int_{\phi>u_p} |\nabla u_p|^p\right)^{1/p'} |\Omega''|^{1/p}\sup|\nabla\phi| + M^q|\Omega''|\sup\phi,\end{aligned}$$

where we have used Hölder's inequality. Set

$$A_p = \left(\int_{\phi>u_p} |\nabla u_p|^p\right)^{1/p},$$

and assume (passing to a subsequence if necessary) that $A_p \to \infty$. Then, according to (4.5), we have for some positive constants C_1 and C_2:

$$A_p^p \le C_1 A_p^{p-1} + C_2 M^q.$$

Dividing by A_p^{p-1}, we obtain $A_p \le C_1 + C_2 M^q A_p^{-(p-1)}$, which implies that $M^q A_p^{-(p-1)} \to \infty$. This is impossible, since it can be easily checked that $M^q A_p^{-(p-1)} = (M^{q/(p-1)} A_p^{-1})^{p-1} \to 0$.

Thus there exists a positive constant C such that $A_p \le C$, which implies

$$\left(\int_{\Omega'} |\nabla u_p|^p\right)^{1/p} \le \left(\int_{\phi > u_p} |\nabla u_p|^p\right)^{1/p} \le C,$$

where we have used that $\Omega' \subset \{\phi > u_p\}$. We now observe that, if we fix $m > N$ and take $p > m$, we have,

$$\left(\int_{\Omega'} |\nabla u_p|^m\right)^{1/m} \le |\Omega'|^{\frac{1}{m}-\frac{1}{p}} \left(\int_{\Omega'} |\nabla u_p|^p\right)^{1/p} \le C.$$

Thanks to the Morrey embedding, we may assert that there exists a positive constant (independent of p) such that (cf. [1], [18]):

$$|u_p(x) - u_p(y)| \le C|x-y|^{1-\frac{N}{m}} \quad x, y \in \Omega'. \tag{4.6}$$

Thus, according to Ascoli-Arzelá theorem, for every sequence $p_n \to \infty$, there exists a subsequence (denoted again $\{p_n\}$) and a function u such that $u_{p_n} \to u$ uniformly in Ω'. By a standard diagonal procedure, and after a repeated choice of subsequences, we can obtain a function $u \in C(\Omega)$ such that $u_{p_n} \to u$ uniformly on compacts of Ω. Passing to the limit in (4.6) we obtain that $u \in C^{N/m}(\Omega)$, and since $m > N$ is arbitrary, we get that $u \in C^\gamma(\Omega)$ for every $\gamma \in (0,1)$.

Now we prove inequality (4.4). Fix $x \in \Omega$. Notice that $B(x, d(x)) \subset \Omega$. Thus, by comparison we arrive at

$$u_{p_n}(y) \le u_{p_n,B}(y) \ \text{ when } y \in B(x, d(x)), \tag{4.7}$$

where $u_{p_n,B}$ is the unique solution to (1.1) in $B(x, d(x))$ with $p = p_n$. For $\delta \in (0, d(x))$ fixed, and thanks to Theorem 2, part 2, we can pass to the limit in (4.7) to arrive at

$$u(y) \le \alpha_0^{\alpha_0} (d(x) - |y - x|)^{-\alpha_0}$$

provided $\delta \le |y - x| \le d(x) - \delta$. Thanks to the continuity of u, we can let $\delta \to 0$ and $y \to x$, to obtain (4.4).

To finish the proof, we show that u is strictly positive in Ω. Observe that the comparison (4.2) in the proof of Lemma 11 remains valid, where $u_{p,B}$ is the unique solution to (4.1), and R, δ are chosen to have $\Omega \subset B(0,R) \setminus B(0,\delta)$. Hence, we may apply Theorem 2, part 2, to obtain that

$$u(x) \ge \alpha_0^{\alpha_0} (R - |x|)^{-\alpha_0},$$

which shows that u is a strictly positive function. □

We finally proceed with the proof of part 2 in Theorem 1. We state it once again for the reader's convenience.

Lemma 14. *Assume $1 < Q < \infty$. Then u_p converges uniformly on compact subsets of Ω to a strictly positive viscosity solution u to*

$$\begin{cases} \max\{-\Delta_\infty u, \ -|\nabla u| + u^Q\} = 0 & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Moreover, if $\alpha_0 = 1/(Q-1)$, then $u(x) \le \alpha_0^{\alpha_0} d(x)^{-\alpha_0}$ in Ω, and there exists $\delta > 0$ such that

$$u(x) = \alpha_0^{\alpha_0} d(x)^{-\alpha_0} \quad \text{in } 0 < d(x) < \delta. \tag{4.8}$$

Furthermore, $u(x)$ is the only positive solution to (1.4) *which verifies*

$$u(x) \sim \alpha_0^{\alpha_0} d(x)^{-\alpha_0} \quad \text{as } d(x) \to 0. \tag{4.9}$$

Proof. Thanks to Lemma 13, for every sequence $p_n \to \infty$, we obtain a subsequence and a strictly positive function $u \in C(\Omega)$ such that $u_{p_n} \to u$ uniformly on compacts of Ω. Moreover, $u \le \alpha_0^{\alpha_0} d(x)^{-\alpha_0}$ in Ω.

We now claim that the limit u is a viscosity solution of (1.4) (cf. [19] for a similar procedure). To prove this, let $\phi \in C^2(\Omega)$, and assume that $u - \phi$ has a strict local maximum at $x_0 \in \Omega$ with $u(x_0) = \phi(x_0)$. We have to check that

$$\max\{-\Delta_\infty \phi(x_0),\ -|\nabla\phi(x_0)| + \phi^Q(x_0)\} \le 0. \tag{4.10}$$

As u_{p_n} converges uniformly to u, each function $u_{p_n} - \phi$ has a local maximum at a point x_n, and $x_n \to x_0$ while $u_{p_n}(x_n) = \phi(x_n) + k_n$ with $k_n \to 0$.

Thanks to Lemma 7, u_{p_n} is a viscosity solution to (3.2), and hence

$$-(p_n - 2)|\nabla\phi|^{p_n-4}\Delta_\infty\phi(x_n) - |\nabla\phi|^{p_n-2}\Delta\phi(x_n) \le -(\phi(x_n) + k_n)^{q_n}. \tag{4.11}$$

Notice that the right-hand side of (4.11) is equal to $-u_{p_n}(x_n)^{q_n} < 0$, and thus $\nabla\phi(x_n) \neq 0$ for every n. Thus we can simplify in (4.11) to obtain

$$-\Delta_\infty\phi(x_n) - \frac{1}{p_n-2}|\nabla\phi|^2\Delta\phi(x_n) \le -\frac{1}{p_n-2}\left[\frac{(\phi+k_n)^{q_n/(p_n-4)}}{|\nabla\phi|}(x_n)\right]^{p_n-4}.$$

As

$$\frac{(\phi+k_n)^{q_n/(p_n-4)}}{|\nabla\phi|}(x_n) \to \frac{\phi^Q}{|\nabla\phi|}(x_0) \tag{4.12}$$

we arrive at

$$\frac{\phi^Q}{|\nabla\phi|}(x_0) \le 1. \tag{4.13}$$

Now, we can pass to the limit and get

$$-\Delta_\infty\phi(x_0) \le 0. \tag{4.14}$$

Inequalities (4.13) and (4.14) give (4.10). This proves that u is a viscosity subsolution.

Now let us check that u is a viscosity supersolution. Notice that the special form of equation (1.4) shows that the argument is not entirely symmetric. Thus let $\phi \in C^2(\Omega)$ and assume that $u - \phi$ has a strict local minimum at x_0 with $u(x_0) = \phi(x_0)$. We have to check that

$$\max\{-\Delta_\infty \phi(x_0),\ -|\nabla\phi(x_0)| + \phi^Q(x_0)\} \ge 0. \tag{4.15}$$

Since u_{p_n} converges uniformly to u, $u_{p_n} - \phi$ has a local minimum at a point x_n with $x_n \to x_0$ and $u_{p_n}(x_n) = \phi(x_n) + k_n$ with $k_n \to 0$.

If $\nabla\phi(x_0) = 0$, then (4.15) is automatically satisfied. If $\nabla\phi(x_0) \neq 0$ then $\nabla\phi(x_n) \neq 0$ for large n, and we can proceed as before and arrive at

$$-\Delta_\infty\phi(x_n) - \frac{1}{p_n-2}|\nabla\phi|^2\Delta\phi(x_n) \ge -\frac{1}{p_n-2}\left[\frac{(\phi+k_n)^{q_n/(p_n-4)}}{|\nabla\phi|}(x_n)\right]^{p_n-4}.$$

Therefore, if $\phi^Q(x_0) < |\nabla\phi(x_0)|$, we would have thanks to (4.12) that $-\Delta_\infty\phi(x_0) \geq 0$. This proves (4.15), and shows that u is indeed a viscosity solution to (1.4).

To show that the convergence is not merely through subsequences, we are next proving (4.9). We only need to check that

$$\liminf_{d(x)\to 0} d(x)^{\alpha_0} u(x) \geq \alpha_0^{\alpha_0}, \tag{4.16}$$

since the reversed inequality is a direct consequence of (4.4) in Lemma 13. To this aim, we are constructing a global subsolution to (1.1).

Choose $\varepsilon \in (0,1)$, and $\delta > 0$ small enough so that the function $d(x)$ is C^2 in $0 < d(x) < \delta$, while $|\nabla d| = 1$ there (cf. [18]). We begin by proving that, for $d(x) < 2\delta$, the function

$$\underline{u}(x) = (1-\varepsilon)^{\frac{1}{q-p+1}} A_p d(x)^{-\alpha_p},$$

where $A_p = (\alpha_p^{p-1}(\alpha_p+1)(p-1))^{\frac{1}{q-p+1}}$, is a subsolution provided that p is large enough (in the rest of the proof we are removing the subindex p in α_p and A_p for brevity). Indeed, it is not hard to see that

$$\Delta_p\underline{u} - \underline{u}^q = (1-\varepsilon)^{\frac{p-1}{q-p+1}} A^{p-1} d^{-(\alpha+1)(p-1)-1} \times \left(\alpha^{p-1}(\alpha+1)(p-1) - \alpha^{p-1} d\Delta d - (1-\varepsilon)A^{q-p+1}\right).$$

Thus thanks to the definition of A, and since $d\Delta d$ is bounded in $d(x) \leq 2\delta$, we obtain that $\underline{u}$ is a subsolution for $p \geq p_0$ (where p_0 depends on δ and ε), provided $d(x) < 2\delta$.

At this point, it is not difficult to see that the function

$$\tilde{u}(x) = \max\{\underline{u} - C, 0\}$$

is also a subsolution for every $C > 0$, which is defined in Ω. If we choose $C = (1-\varepsilon)^{\frac{1}{q-p+1}} A\delta^{-\alpha}$, we have $\tilde{u}(x) = 0$ for $d(x) \geq \delta$. Hence we can use the comparison principle once again to deduce that $u_p(x) \geq \tilde{u}(x)$ in Ω, for $p \geq p_0$.

In particular, letting $p = p_n$ and passing to the limit, we arrive at

$$u(x) \geq \alpha_0^{\alpha_0}(d(x)^{-\alpha_0} - \delta^{-\alpha_0}) \quad \text{for } d(x) < \delta.$$

This implies (4.16), and thus (4.8).

We now remark that (1.4) admits a unique solution verifying (4.8), thanks to Theorem 10. Thus, since for every sequence $p_n \to \infty$ the limit of u_{p_n} will be this solution, we deduce that $u_p \to u$ as $p \to \infty$.

We finally prove (4.9), that is, $u = \alpha_0^{\alpha_0} d(x)^{-\alpha_0}$ in a neighbourhood of $\partial\Omega$. Thanks to (4.4), we only need to check that $u(x) \geq \alpha_0^{\alpha_0} d(x)^{-\alpha_0}$ in a neighbourhood of $\partial\Omega$.

Let $x_0 \in \partial\Omega$ and choose an annulus $A = \{x \in \mathbb{R}^N : R_1 < |x - \tilde{x}_0| < R_2\}$ such that $\Omega \subset A$ and A is tangent to $\partial\Omega$ at x_0 (in particular $\tilde{x}_0 = x_0 + R_1\nu(x_0)$, where $\nu(x_0)$ is the outward unit normal at x_0, so that $|x_0 - \tilde{x}_0| = R_1$). Let $u_{p,A}$ be the unique solution to

$$\begin{cases} \Delta_p u = u^q & \text{in } A \\ u = +\infty & \text{on } \partial A. \end{cases}$$

It follows by comparison that $u_p \geq u_{p,A}$ in Ω. Hence, passing to the limit we arrive at $u \geq u_A$ in Ω, where u_A is the unique solution to (1.4) in A. However, thanks to Remark 1 (c) and uniqueness, we get that $u_A = \alpha_0^{\alpha_0} d_A(x)^{-\alpha_0}$, where $d_A(x) = \mathrm{dist}(x, \partial A)$. Since $d_A(x) = d(x)$ for points of the form $x_0 - t\nu(x_0)$ and small positive t, we have shown that $u(x) \geq \alpha_0^{\alpha_0} d(x)$ if $x = x_0 - t\nu(x_0)$ for small positive t. A compactness argument shows that there exists a positive δ so that $u(x) \geq \alpha_0^{\alpha_0} d(x)^{-\alpha_0}$ if $0 < d(x) < \delta$. This completes the proof. □

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